LEGENDRE'S POLYNOMIAL

GENERATING FUNCTIONS FOR LEGENDRE :-

The function

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, z^n \qquad \text{——— (1)}$$

is known as generating function for Legendre's polynomial.

PROOF :-

$$(1-2xz+z^2)^{-1/2} = \{1 - z(2x-z)\}^{-1/2} \; *$$

$$= 1 + \frac{1}{2}\cdot z(2x-z) + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!} z^2(2x-z)^2 + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!} z^3(2x-z)^3 + \dots$$

$$\dots + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\frac{(2n-5)}{2}}{(n-2)!} z^{n-2}(2x-z)^{n-2} + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\frac{(2n-3)}{2}}{(n-1)!} z^{n-1}(2x-z)^{n-1}$$

$$+ \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\frac{(2n-1)}{2}}{n!} z^n(2x-z)^n + \dots$$

$$= 1 + \frac{1}{2} z(2x-z) + \frac{1\cdot 3}{2\cdot 4} z^2(2x-z)^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} z^3(2x-z)^3$$

$$\dots + \frac{1\cdot 3\cdot 5\cdots(2n-5)}{2\cdot 4\cdot 6\cdots(2n-4)} z^{n-2}(2x-z)^{n-2} + \frac{1\cdot 3\cdot 5\cdots(2n-3)}{2\cdot 4\cdot 6\cdots(2n-2)} z^{n-1}(2x-z)^{n-1}$$

$$+ \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n} z^n(2x-z)^n + \dots \qquad \text{——— (2)}$$

⇒ Now In order to find co-efficient of z^n in whole series we must have to expand $(2x-z)$ with respect to their corresponding power in nth, $(n-1)^{th}$ and $(n-2)^{th}$ term in equation (2);

For nth term;

$$\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n} z^n(2x-z)^n = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n(1\cdot 2\cdot 3\cdots n)} z^n\left[(2x)^n - (2x)^{n-1}z + \dots\right] \; *$$

$$= \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\, n!} 2^n x^n z^n - \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\, n!} 2^{n-1}x^{n-1}z^{n+1} + \dots$$

$*$ Identity used above -

(a) $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \dots$

(b) $(x-a)^n = {}^nC_0\, x^{n-0}a^0 + {}^nC_1\, x^{n-1}a^1 + {}^nC_2\, x^{n-2}a^2 + \dots$

Hence co-efficient of z^n in n^{th} term will be

$$\frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2^n\, n!}\, 2^n x^n = \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{n!}\, x^n \qquad \text{—— (3a)}$$

Similarly for $(n-1)^{th}$ term;

$$\frac{1\cdot 3\cdot 5 \cdots (2n-3)}{2\cdot 4\cdot 6 \cdots (2n-2)}\, z^{n-1} (2x-z)^{n-1}$$

$$= \frac{1\cdot 3\cdot 5 \cdots (2n-3)}{2\cdot 4\cdot 6 \cdots (2n-2)}\, z^{n-1}\left[(2x)^{n-1} - {}^{n-1}C_1 (2x)^{n-2} z + {}^{n-1}C_2 (2x)^{n-3} z^2 + \cdots\right]$$

$$= \frac{1\cdot 3\cdot 5 \cdots (2n-3)}{2\cdot 4\cdot 6 \cdots (2n-2)}\, z^{n-1} (2x)^{n-1} - \frac{1\cdot 3\cdot 5 \cdots (2n-3)}{2\cdot 4\cdot 6 \cdots (2n-2)} (n-1)(2x)^{n-2} z^n + \cdots$$

Hence co-efficient of z^n in $(n-1)^{th}$ term is clearly,

$$\frac{1\cdot 3\cdot 5 \cdots (2n-3)}{2\cdot 4\cdot 6 \cdots (2n-2)} (n-1)(2x)^{n-2} = -\frac{1\cdot 3\cdot 5 \cdots (2n-3)(2n-1)\, n(n-1)}{2\cdot 4\cdot 6 \cdots (2n-2)(2n-1)\, n}\, 2^{n-2}\cdot x^{n-2}$$

$$= -\frac{1\cdot 3\cdot 5 \cdots (2n-3)(2n-1)}{2^{n-1}\, n!}\cdot \frac{n(n-1)}{(2n-1)}\cdot x^{n-2}$$

$$= -\frac{1\cdot 3\cdot 5 \cdots (2n-1)}{n!}\cdot \frac{n(n-1)}{2(2n-1)}\, x^{n-2} \qquad \text{—— (3b)}$$

Again for $(n-2)^{th}$ term co-efficient of z^n is given by,

$$\frac{1\cdot 3\cdot 5 \cdots (2n-1)}{n!}\cdot \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot (2n-1)(2n-3)}\, x^{n-4} \qquad \text{—— (3c)}$$

Proceeding farther as above, we conclude that from (3a), (3b) and (3c) .... that the co-efficient of z^n in the expansion of (2) is,

$$\frac{1\cdot 3\cdot 5 \cdots (2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot (2n-1)(2n-3)} x^{n-4} - \cdots\right]$$

which is $P_n(x)$.

Thus we can say that in the expansion of $(1-2xz+z^2)^{-1/2}$ the co-efficient of $z, z^2, z^3, \ldots z^n$ etc. will be $P_1(x), P_2(x) \ldots$ etc.

Hence we can write,

$$(1-2xz+z^2)^{-1/2} = 1 + zP_1(x) + z^2 P_2(x) + \cdots + z^n P_n(x) + \cdots$$

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, z^n$$