LAGUERRE'S POLYNOMIAL

Questions:-

(i) Show that $L_n(x)$ is the co-efficient of z^n in the expression of $\left(\frac{e^{-xz/1-z}}{1-z}\right)$.

(ii) prove that $\frac{e^{-xz/1-z}}{1-z} = \sum_{n=0}^{\infty} L_n(x) z^n$

Where, $L_n(x) = \sum_{n=0}^{\infty} \frac{(-1)^r n!\, x^r}{(r!)^2 (n-r)!}$

(iii) Derive an expres. For generating function for the Laguerre's polynomial.

GENERATING FUNCTION FOR LAGUERR'E :-

The Generating function for Laguerre's Polynomial is,

$$f(x,z) = \frac{1}{(1-z)} e^{-xz/1-z} = \sum_{n=0}^{\infty} L_n(x) z^n$$

PROOF :-

$$\frac{1}{(1-z)} e^{-xz/1-z} = \frac{1}{(1-z)} \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{-xz}{1-z}\right)^r \quad *$$

$$= \frac{1}{(1-z)} \sum_{r=0}^{\infty} \frac{1}{r!} (-1)^r \frac{x^r z^r}{(1-z)^r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{x^r z^r}{(1-z)^{r+1}}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} x^r z^r (1-z)^{-(r+1)} \quad *$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r x^r z^r}{r!} \left[1 + (r+1)z + \frac{(r+1)(r+2)}{2!} z^2 + \frac{(r+1)(r+2)(r+3)}{3!} z^3 + \cdots\right]$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!} \left[z^r + (r+1) z^{r+1} + \frac{(r+1)(r+2)}{2!} z^{r+2} + \frac{(r+1)(r+2)(r+3)}{3!} z^{r+3} + \cdots\right]$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!} \left[\frac{z^r\, r!}{0!\, r!} + \frac{r!\,(r+1)\, z^{r+1}}{1!\, r!} + \frac{(r+1)(r+2) r!\, z^{r+2}}{2!\, r!} + \cdots\right] *$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!} \left[\frac{(r+0)!\, z^{r+0}}{0!\, r!} + \frac{(r+1)!\, z^{r+1}}{1!\, r!} + \frac{(r+2)!\, z^{r+2}}{2!\, r!} + \cdots\right]$$

(i) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{r=0}^{\infty} \frac{x^r}{r!}$

(ii) Identity - $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n+2)}{3!} x^3 + \cdots$

(iii) $(r+1)! = (r+1)\, r!$

$(r+2)! = (r+2)(r+1)\, r!$

(iv) $0! = 1$, $1! = 1$

$$\frac{e^{-xz/1-z}}{(1-z)} = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!} \left[\sum_{s=0}^{\infty} \frac{(r+s)!\, z^{r+s}}{s!\; r!} \right]$$

For $r+s=n$, so that, $s=n-r$

we get,

$$\frac{e^{-xz/1-z}}{(1-z)} = \overset{*}{\sum_{r=0}^{n}} \frac{(-1)^r x^r}{r!} \sum_{n=0}^{\infty} \frac{n!\, z^n}{(n-r)!\, r!}$$

*Here $n-r \geq 0$
or
$n \geq r$
Hence limit of r starts from 0 and go to n. not to ∞.

Arranging above equation,

$$\frac{e^{-xz/1-z}}{(1-z)} = \sum_{n=0}^{\infty} \left[\sum_{r=0}^{n} (-1)^r \frac{x^r\, n!}{(r!)^2\, (n-r)!} \right] z^n$$

$$\frac{e^{-xz/1-z}}{(1-z)} = \sum_{n=0}^{\infty} L_n(x)\, z^n \qquad \text{——— (1)}$$

where, $L_n(x) = \sum_{r=0}^{n} (-1)^r \frac{x^r\, n!}{(r!)^2\, (n-r)!}$

Thus the function $\frac{e^{-xz/1-z}}{1-z}$ generates all the Laguerre's polynomial and hence it is Called Generating function of Laguerre's polynomial.